Kerala School of Mathematics
Frontiers in Mathematics
Quantum doubly stochastic maps and their asymptotics
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Remark: All the inner products considered are anti-linear in the first variable and linear in the second variable. For $d \geq 1$, $\mathbb{M}_d$ denotes the C^*-algebra of all $d \times d$ complex matrices considered as the space of linear maps on $\mathbb{C}^d$. Standard inner product is considered on $\mathbb{C}^d$ unless some other inner product is explicitly mentioned. All projections considered are orthogonal projections.

Lecture 1: Quantum Probability

(1) Show that any state (positive linear functional mapping identity to 1) on $\mathbb{M}_d$ is of the form

$$X \mapsto \text{ trace } (\rho X),\ X \in \mathbb{M}_d$$

for some *density* matrix ρ. (ρ is a density matrix if $\rho \geq 0$ and trace $(\rho) = 1$.

(2) Let $\mathcal{S}_d$ be the set of all states on $\mathbb{M}_d$. Show that $\mathcal{S}_d$ is a convex, compact set whose extreme points are given by maps of the form

$$X \mapsto \langle u, Xu\rangle,\ X \in \mathbb{M}_d,$$

for unit vectors $u \in \mathbb{C}^d$. These are known as *pure* states.

(3) An observable (self-adjoint matrix) is called a spin observable if the only values taken by it are ± 1. Show that A is a spin-observable if and only if $A = 2P - I$, where P is a projection.

(4) Give an example of a covariance matrix of three spin observables $C = [c_{ij}]_{1\leq i,j\leq 3}$, which violates Bell- inequality:

$$1 - c_{12} \geq |c_{13} - c_{23}|.$$

(Ans: See Example 5.4 of [KRP])

Lecture 2: Quantum doubly stochastic maps.

(1) Prove Birkhoff von Neumann theorem. What is the similar theorem for stochastic maps?

(2) Show that convex combinations and compositions of completely maps are completely positive.

(3) (Schur maps) Let $A \in M_d$. Define $\sigma_A : \mathbb{M}_d \to \mathbb{M}_d$ by

$$\sigma_A(X) = A \circ X,\ \forall X \in \mathbb{M}_d,$$

where $\circ$ denotes the Schur map. Show that σ_A is completely positive if and only if A is positive and it is unital and trace preserving if and only if the diagonal entries of A are equal to 1. If A is positive, obtain the Choi-Kraus decomposition of σ_A.

(4) Let $\tau : \mathbb{M}_d \to \mathbb{M}_d$ be a completely positive map. Show that τ is contractive if and only if $\tau(I) \leq I$.

(5) Show that $\mathbb{M}_d$ with

$$\langle X, Y\rangle := \text{ trace } (X^*Y),\ X, Y \in \mathbb{M}_d$$

is a Hilbert space. This inner product is known as the Hilbert-Schmidt inner product. Let $\alpha : \mathbb{M}_d \to \mathbb{M}_d$ be the linear map defined by

$$\alpha(X) = AXB,\ \forall X \in \mathbb{M}_d,$$

for fixed A, B in $\mathbb{M}_d$. Show that α^* is given by

$$\alpha(X) = A^* X B^*, \ \forall X \in \mathbb{M}_d.$$

(6) Defind $\delta_d : \mathbb{M}_d \to \mathbb{M}_d$ by

$$\delta_d(X) = \frac{1}{d} \text{ trace } (X) I, \ \forall X \in \mathbb{M}_d.$$

Show that δ_d is a mixed unitary quantum channel.

(7) Show that the *Holevo-Werner channel,* $\tau : \mathbb{M}_3 \to \mathbb{M}_3$ defined by

$$\tau(X) = \frac{1}{2}(\text{ trace } (X) I - X^t), \ X \in \mathbb{M}_3$$

is a unital quantum channel, which is not a mixed unitary.

Lecture 3: Asymptotics of quantum doubly stochastic semigroups.

(1) Let $J_n(\lambda)$ denote a Jordan block of $n \times n$ with eigenvalue λ (upper triangular matrix with diagonal entries equal to λ, super diagonal entries equal to 1 and all other entries equal to 0. Show that $\lim_{m\to\infty}(J_n(\lambda)) = 0$ if and only if $|\lambda| < 1$.

(2) Suppose $\tau : \mathbb{M}_d \to \mathbb{M}_d$ is a unital quantum channel. Show that τ is contractive in Hilbert-Schmidt norm.

(3) Consider a unital quantum channel $\tau : \mathbb{M}_d \to \mathbb{M}_d$ given by

$$\tau(X) = \sum_{k=1}^{p} A_k X A_k^*, \ \forall X \in \mathbb{M}_d.$$

Show that Y in $\mathbb{M}_d$ is a fixed point of τ if and only if

$$Y A_k = A_k Y$$

for all k. Show that the space of fixed points of τ is a C^*-subalgebra of $\mathbb{M}_d$. Similarly, show that the *peripheral space*:

$$\mathcal{P}_\tau := \text{ span } \{Y : \tau(Y) = \lambda Y, \ |\lambda| = 1\}$$

is a C^*-subalgebra of $\mathbb{M}_d$

(4) Let $\mathcal{A}$ be a unital C^*-subalgebra of $\mathbb{M}_d$. Then a completely positive map $E : \mathbb{M}_d \to \mathbb{M}_d$ such that the range of E is $\mathcal{A}$ and $E(Y) = Y$ for every Y in $\mathcal{A}$ is called a *conditional expectation map* onto $\mathcal{A}$. Suppose E is such a conditional expectation map show that

$$E(AYB) = AE(Y)B, \ \forall A, B \in \mathcal{A}, Y \in \mathbb{M}_d.$$

References

[KRP] K. R. Parthasarathy, An Introduction to quantum stochastic Calculus, Birkhauser.

[RB] B. V. Rajarama Bhat and R. Devendra, Regions of mixed unitarity for semigroups of unital quantum channels, arXiv:2512.23598v2 dated 7 Feb. 2026.